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The interpretation of Paul's determination not to know anything among men save Jesus Christ and him crucified that makes him mean to refer to a series of historic facts eviscerates it of all real content. In the creation was the Calvary of Deity. The cross is thus the background plan of the universe. To know the cross from this higher standpoint is to know all there is to know; there is nothing beyond this. The cross was the symbol of a profound mystery which opened up the heart of Deity himself to the gaze of the world. The divine sufferer was God himself, who in creating the universe sacrificed himself for it. The cross, therefore, represents the greatest of all sacrifices, not something that happened once and once for all, but something that is eternal and timeless—the sacrifice of God in and for his own creation that could not be unless he poured his own life into it and restricted himself within the forms of matter. "Confessedly great is the mystery of godliness." Unthinkable in its magnitude is this sacrifice, for it means nothing less than the identification of the infinite with the finite in its lowest forms. Here is the profoundest mystery open to human contemplation to speak of which is possible only in forms of symbol and parable. The literal truth is too vast, too mysterious, too sublime, to be made known to human comprehension. It is the mystery before which angels, we are told, veil their faces; and to gain a single glimpse of it one may well surrender all other knowledge and determine, as Paul did, to know nothing else. Here is the oldest form of the Christian faith. The story of Jesus is the parable of this infinitely larger truth. It is the symbol of the Lamb slain from the foundation of the world, that is, prior to human history, the emblem of divine body and blood voluntarily sacrificed in outward physical nature and entombed in the lower consciousness of man. It was the claim of the second century Gnostics that Christianity was none other than the consummation of the inner doctrine of the mystery institutions of all the nations; and it is this interpretation of the Gospel story which is set forth in the apocryphal Gospels and Acts. The end of them all was the revelation of the mystery of man which is none other than the mystery of Christ.

Dundee, Scotland. K. C. Anderson.

HAMILTON'S HODOGRAPH.

In Mach's *Mechanics* the space devoted to the hodograph of Sir William Rowan Hamilton is barely a page, half of which is taken up by the diagram; the diagram, too, is not drawn strictly according to the construction of Hamilton, but according to the usual manner of present-day text-books. This seems to me to be something of an injustice to an exceedingly brilliant piece of work; for from it and Hamilton's theorem of the isochronism of the circular hodograph can be deduced the more important properties of motion in orbits described to a center of force according to the Newtonian law, Lambert's theorem for the area of an elliptic sector in terms of the bounding *radii vectores* and the chord of the sector, and Euler's corresponding expression for the area of a parabolic sector. The proofs are exceedingly simple and demand very little knowledge of the properties of conics, and no calculus or coordinate geometry at all. For other interesting and elegant applications of the hodograph, reference should be made to Tait's *Quaternions* and papers in the *Proc. R. S. E.*, and to Tait and Steele's *Dynamics of a Particle.* But the matter which follows should, I think, be sufficient to corroborate the opinion expressed above.

I. The first use of the curve is ascribed to Bradley,[1] and it is probably his definition that is generally given in elementary text-books, where its only use is to obtain the acceleration towards the center of a particle moving in a circle with uniform speed.

DEF. 1. If a point be in motion with any velocity in any orbit, and if at any instant a line be drawn from a fixed point representing on some chosen scale the velocity of the point at the instant in magnitude and direction, the locus of its end is the "hodograph" of the motion.

Not even in the particular case of motion in a circle does this, the usual definition, so readily demonstrate the connection between the hodograph and the orbit as the definition originally given by Hamilton in a paper communicated to the Royal Irish Academy in 1846,[2] which is as follows:

DEF. 2. "If, in an orbit *round a center of force,* there be taken on the perpendicular from the center on the tangent at each point a length equal to the velocity at that point of the orbit, the extremities of these lengths will trace out the hodograph."

Note. The use of the word "equal" does not introduce any difficulty, unless we attempt to verify the formulas obtained by reference to the theory of dimensional units. It is to be noted that the definition of Hamilton is more specialized than the other, referring only to central orbits.

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[1] Sir Robert Ball, Article on "Gravitation," *Encyc. Brit.*

[2] *Proc. Roy. Ir. Acad.,* 1847.

II. Newton showed that for any central force the areas swept out by the radius vector are proportional to the times.[3] The following is a simple proof of this theorem.

If O is the center of force, and P, Q, R points on the orbit separated by *very small unit intervals* of time, and MQN, PM, RN are drawn perpendicular to OQ, MQ, QN respectively; then MQ, QN measure the velocities transverse to OQ, before and after the position Q.

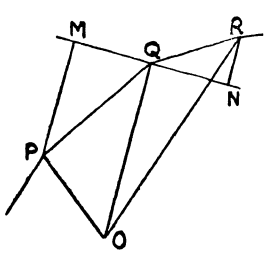

Fig. 1.

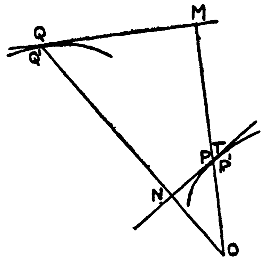

Fig. 2.

Since the force is central, MQ = QN,

$$\therefore \triangle \mathrm{OPQ} = \triangle \mathrm{OQR};$$

and hence the proposition follows immediately.

It follows from Newton's theorem that, if the rate at which the area is swept out is denoted by $\frac{1}{2}h$, and the perpendicular from O on the tangent at P by p, then $pv = h$.

Hence, in Fig. 2, ON.OQ = h, and Q is the image of N in a circle whose radius is $\sqrt{h}$; also, if P′, Q′ are near points to P, Q, and the tangents at P, P′ meet in T, then Q, Q′ are the poles of two lines passing through T, and therefore QQ′ is the polar of T. That is, the tangent at Q is the polar of P.

The hodograph is therefore the polar reciprocal of the orbit.

It is also evident that OM = h/OP; or, in other words, the product of the length of the perpendicular from the center on the tangent to the hodograph and the length of the corresponding radius vector of the orbit is constant (= h).

Again, since the tangent at Q is the direction of the velocity of Q in the hodograph, this velocity is perpendicular to the radius vector. The hodograph given by the usual text-book definition is therefore identical with that given by Hamilton's definition, when turned through a right angle.

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[3] Newton, *Principia,* Book I, Prop. 1.

Further, since OQ, OQ′ represent the velocities at P, P′ turned through a right angle, it follows that QQ′ represents the change of velocity between P and P′ when turned through a right angle. Hence the radial acceleration of P is measured by the velocity of Q in the hodograph.

IV. The angle between the normals at QQ′ is equal to the angle between the radii OP, OP′.

Hence in Fig. 3 we have

$$QQ'/\rho = PN/r = 2\Delta OPP'/r^2;$$

and if the acceleration in the orbit is denoted by f, then

$$f/\rho = h/r^2.$$

This formula for the radius of curvature of the hodograph solves immediately the case

$$f \propto 1/r^2.$$

For, if $f \propto 1/r^2$, ρ is constant, and the hodograph is therefore a circle, and its polar reciprocal is a conic section with the pole as a focus; conversely, if the orbit is conic, with a center of force at a focus, the polar reciprocal is a circle, ρ is constant, and the law of force is $f \propto 1/r^2$.

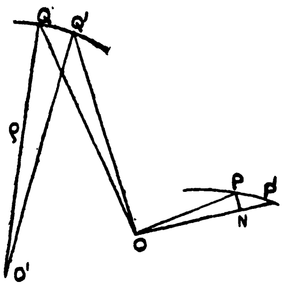

Fig. 3.

The feet of the perpendiculars from the focus to the tangents to the orbit lie on the auxiliary circle (the tangent at the vertex in the case of the parabola). Hence the hodograph is an inverse of the auxiliary circle or the tangent at the vertex in the case of the parabola. It follows that the focus is within, on, or without the hodograph, according as the orbit is an ellipse, parabola, or hyperbola.

On account of the fact that the hodograph of an orbit described under the Newtonian law, $f \propto 1/r^2$, is a circle, Hamilton designated the law as the "Law of the Circular Hodograph."

V. If the law is $f = \mu/r^2$, the diameter of the hodograph is easily seen to be $2\mu/h$. Also, the diameter through O of the hodograph corresponds to the diameter *aa′* of the orbit and is similarly divided at O.

In Fig. 4, let A, B, A′, B′ correspond with the tangents at a, b, a', b', and therefore OA, OB, OA′, OB′ represent the velocities at the points a, b, a', b'.

Now, $aa' = \mathrm{O}a + \mathrm{O}a' = h(1/\mathrm{OA} + 1/\mathrm{OA'}) = h.\mathrm{AA'}/\mathrm{OA}.\mathrm{OA'}$.

If, then $aa' = 2a$, we have $a = hp/\mathrm{OB}^2$; i. e., $\mathrm{OB}^2 = hp/a = \mu/a$.

Therefore, the velocity at the end of the minor axis of the eclipse

$$= \sqrt{\mu/a}.$$

The velocity at any point of the orbit is now at once obtainable.

Case I. When O is within the hodograph, as in Fig. 5.

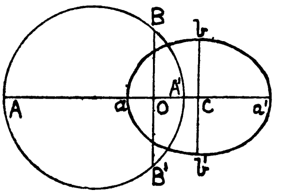

Fig. 4.

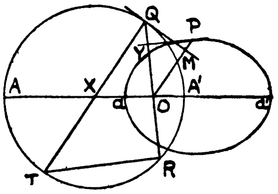

Fig. 5.

If QO and QX meet the hodograph a second time in R and T, the triangles OQM, QTR are similar;

$$\mathrm{QT} : \mathrm{OQ} = \mathrm{QR} : \mathrm{OM},$$

$$\therefore 2p.\mathrm{OM} = \mathrm{OQ}.\mathrm{QR} = \mathrm{OQ}^2 + \mathrm{QO}.\mathrm{OR} = \mathrm{OQ}^2 + \mathrm{AO}.\mathrm{OA'};$$

$$\therefore \mathrm{OQ}^2 = 2ph/r - ph/a \qquad \text{(from § IV)}$$

$$\text{i. e., } \mathrm{V}^2 = \mu(2/r - 1/a).$$

Case II. When O is on the hodograph, $\mathrm{V}^2 = 2\mu/r$.

Case III. When O is without, $\mathrm{V}^2 = \mu(2/r + 1/a)$.

VI. A most remarkable theorem connected with the hodograph was communicated to the Royal Irish Academy in March, 1847, without demonstration,[4] by Hamilton; he however furnished a proof by quaternions in 1853.[5] The following simple geometrical proof arises from hints given by Hamilton's proof.[6]

Theorem. "If two circular hodographs, which have a common chord passing or tending through a common center of force, be both cut at right angles by a third circle, the times of hodographically describing the intercepted arcs will be equal."

[4] Cayley's report to the Brit. Assoc., 1862.

[5] Hamilton, *Lectures on Quaternions,* 1853.

[6] Professor Blyth, Article on "Hodograph" in *Encyc. Brit.*

If ρ is the radius of the hodographic circle in Fig. 6, and the law of force is $f = \mu/r^2$, then, by § IV, we have

$$\rho h = fr^2 = \mu \text{ (a constant)};$$

also, by § III, we have $r.\mathrm{OM} = h$, and therefore

$$t = \mathrm{PQ}/f = \mu.\mathrm{PQ}/\rho^2.\mathrm{OM}^2.$$

Now the angles PXQ, RVR′ are equal and small,
$\therefore \mathrm{PQ} = \rho.\sin \mathrm{RVR}' = \rho.\mathrm{RR}'.\sin \mathrm{VRR}'/\mathrm{VR}' = \rho.\mathrm{RR}'.\mathrm{OM}/\mathrm{OR}.\mathrm{PR}$
ultimately; and, from the similar triangles, XPL, ROM,

$$\rho.\mathrm{OM} = \mathrm{OR}.\mathrm{PL};$$

$$t = \mu.\mathrm{RR}'/\mathrm{OR}^2.\mathrm{PL}.\mathrm{PR} = \lambda/\mathrm{PL}.$$

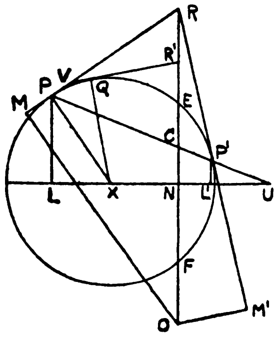

Fig. 6.

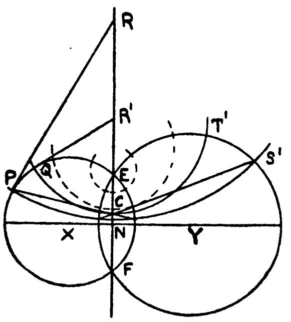

Fig. 7.

Similarly, for the description of the arc P′Q′, we have $t = \lambda/\mathrm{P'Q'}$,

$$t + t' = \lambda(1/\mathrm{PL} + 1/\mathrm{P'L'}) = 2\lambda/\mathrm{CN}.$$

But any two circles, intersecting in EF, are cut orthogonally by a system of circles having their centers on EF produced in either direction. If we start with the orthogonal PSP′S′ in Fig. 7, C is the radical center of the three circles whose centers are R, X, Y; and hence, since the times of describing arc PQ + arc P′Q′ and arc ST + arc S′T′ are each equal to 2λ/CN, these times are equal. Therefore, by "filling up" the arcs PP′, SS′ with a system of orthogonals, it follows that the times of description of the whole arcs PP′ and SS′ are equal.

VII. Let us consider the whole system of coaxal circles passing through E and F, and their corresponding orbits.

If the diameters of the hodographs passing through O are drawn, the products of the segments into which they are divided at O are respectively equal to μ/a_1, μ/a_2, $\mu/a_3, \ldots$, where a_1, a_2, a_3, ...are the semi-major axes of the orbits, when O does not coincide with either E or F. Hence, since O is on the radical axis of the system, it follows that the major axes are all equal.

If O coincides with E or F, and $4a$ is the latus-rectum of one of the parabolic orbits, $2\rho . a = h$, or $a = h^2/2\mu$.

Again, any one of the hodographs can be completely traced out by a set of orthogonal circles whose centers traverse the whole of the radical axis except the part EF intercepted within the hodographs. Therefore, by Hamilton's theorem, the periodic times are all equal, for each separate position of the point O.

Further, if the orbits are supposed to have their major axes all in one straight line, so that O is the middle point of EF, one of the orbits is the circle of radius a, which also passes through the ends of the minor axes of all the orbits. The radius of its hodograph is equal to the velocity at the end of a minor axis of the general orbit of the system, and is equal to $\sqrt{\mu/a}$. Also, the velocity in this hodograph is equal to the acceleration in the circular orbit, i. e., is equal to μ/a^2. Therefore, we have

$$\text{Periodic Time} = T = 2\pi\sqrt{\mu/a} \div \mu/a^2 = 2\pi\sqrt{a^3/\mu},$$

for the whole system of orbits.

VIII. Hamilton pointed out that his theorem of isochronism was the same as Lambert's theorem. This is not generally taken to mean the detailed elliptic quadrature theorem as given in Williamson's *Calculus* and elsewhere, but the central force theorem thus enunciated:

In elliptic orbits of equal major axes, described in the same periodic time, under the action of a central force to a focus, the times of describing any sectors are the same, if the chords of the sectors and also the sum of the bounding radii vectores are the same.

A geometrical proof of the theorem in this form was given by H. R. Droop,[7] who pointed out that Hamilton had not, to his knowledge, previously published any demonstration of it; a short analytical proof of Hamilton's theorem is to be found in Tait and Steele's *Dynamics of a Particle,* with the remark, "It will readily be seen that this is in substance the same as Lambert's theorem." On the other hand, Cayley gave an analytical proof of Hamilton's theorem by assuming that of Lambert.[8]

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[7] *Quart. Journ. Math.,* Vol. I, pp. 374-378.

[8] *Phil. Mag.,* 1857, pp. 427-430.

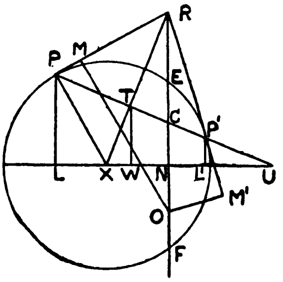

Fig. 8.

The following simple demonstration is practically the same as that of Droop, with the slight improvement that the connection, with lines in the hodograph, of the sum of the *radii vectores* and of the chord of the sector is plainly shown.

In Fig. 8, since the triangles XPL, ROM are similar, as are also XP'L', ROM',

$$\therefore r+r' = h/\mathrm{OM} + h/\mathrm{OM}',$$
$$= \rho h(1/\mathrm{PL} + 1/\mathrm{P'L'})/\mathrm{OR},$$
$$= 2\mu/\mathrm{OR}.\mathrm{CN};$$

hence $r+r'$ is constant.

Again, if δ is the chord, p the perpendicular on it from the focus, and θ the angle between the radii in the orbit which is equal to the angle at the center of the hodograph, then, since R is the pole of the chord, $p.\mathrm{OR} = h$;

$$\therefore \delta = rr' \sin\theta/p = h\mathrm{OR} \sin\theta/\mathrm{OM}.\mathrm{OM}' = \mu\rho \sin\theta/\mathrm{OR}.\mathrm{PL}.\mathrm{P'L'}.$$

But $\mathrm{PL}.\mathrm{P'L'} = \mathrm{TW}.\mathrm{CN} = \mathrm{CN}.\mathrm{NR}.\mathrm{XT}/\mathrm{XR}$,

$$\text{and } \rho \sin\theta = 2\mathrm{XT}.\mathrm{PR}/\mathrm{XR} = 2\mathrm{PR}.\mathrm{XT}/\mathrm{XR};$$

$$\therefore \delta = 2\mu.\mathrm{PR}/\mathrm{CR}.\mathrm{CN}.\mathrm{NR} = \text{a constant.}$$

Note. $r+r' : \delta = \mathrm{NR} : \mathrm{PR}$.

IX. Although no loss of generality is introduced in the last part of § VII by the assumption that the axes of the orbits all lie in one straight line, there will be a loss of generality if we attempt to work out the time of description for a sector on the same assumption; for the inclusion of the circle whose center is O determines the value of $r+r'$ to be equal to $2a$. In this case the chord must be either parallel to the major axis of any orbit or a diameter of it; and the latter is excluded from consideration by Hamilton's theorem, since it is impossible for an orthogonal circle to cut a hodographic circle in points on opposite sides of the diameter perpendicular to the radical axis. When, however, O is not taken to be the middle point of EF, all positions of the *radii vectores* are included; and one of these positions is that in which the chord is parallel to the minor axis.

From Fig. 9, it is evident that area of $p\mathrm{C}q\mathrm{S} = \mathrm{CS}.p\mathrm{N}$, and area of $p\mathrm{C}q\mathrm{A} = a^2 \sin^{-1} y/b$;

$$\therefore \text{ area of PSQA} = ab\,(\sin^{-1} y/b - ey/b).$$

Similarly, in the case of the parabola, it is seen from Fig. 10 that, since area of segment PAQ $= \tfrac{4}{3}$ area of $\propto$PAQ,

$$\therefore \text{ area of sector PSQA} = y(x+3a)/3.$$

X. From the results obtained in § IX, we can show in a very simple manner that not only does Hamilton's theorem contain that of Lambert in its detailed form, but also the corresponding theorem of Euler for a parabolic sector.

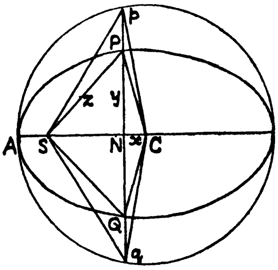

Fig. 9.

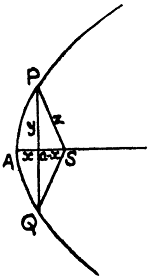

Fig. 10.

LAMBERT'S THEOREM:

Assume $\phi - \phi' = 2\sin^{-1} y/b$ and $\sin\phi = \sin\phi' = 2ey/b$;

then, since $\sin\tfrac{1}{2}(\phi-\phi') = y/b$, $\therefore \cos\tfrac{1}{2}(\phi-\phi') = x/a$;

also since $2\sin\tfrac{1}{2}(\phi-\phi')\cos\tfrac{1}{2}(\phi+\phi') = 2ey/b$,

$$\therefore \cos\tfrac{1}{2}(\phi+\phi') = e, \quad \therefore \sin\tfrac{1}{2}(\phi+\phi') = b/a;$$

$$\cos\phi' + \cos\phi = 2\cos\tfrac{1}{2}(\phi+\phi')\cos\tfrac{1}{2}(\phi-\phi') = 2ex/a,$$

$$\cos\phi' - \cos\phi = 2\sin\tfrac{1}{2}(\phi+\phi')\sin\tfrac{1}{2}(\phi-\phi') = 2y/a.$$

Now, referring to Fig. 9, if SP is equal to z, $a - ex = z$, and therefore $ex/a = 1 - z/a$; hence

$$\cos\phi = 1 - (z+y)/a, \text{ and } \cos\phi' = 1 - (z-y)/a;$$

and these expressions only contain z, y, a, which are equivalent to $\tfrac{1}{2}(r+r')$, $\tfrac{1}{2}\delta$, a; and these are invariable. The time of description being also invariable as well as the period for the whole orbit, we can say that in general,

Area of elliptic sector $= \tfrac{1}{2}ab\,[\phi - \phi' - (\sin\phi - \sin\phi')]$, where

$$4\sin^2\tfrac{1}{2}\phi=(r+r'+\delta)/a \text{ and } 4\sin^2\tfrac{1}{2}\phi=(r+r'-\delta)/a.$$

EULER'S THEOREM:

By VII, $h^2=2a\mu$; and the area described $=ht=\sqrt{2a\mu t}$; hence

$$t=y(x+3a)/3\sqrt{2a}\mu=2\sqrt{x}(x+3a)/3\sqrt{2\mu};$$

and, assuming $\lambda^2=z+y$, and $\mu^2=z-y$, we have

$$2(a+x)=2z=\lambda^2+\mu^2$$

$$2(a-x)=2\sqrt{(z^2-y^2)}=2\lambda\mu.$$

Therefore, $2\sqrt{x}=\lambda-\mu$, $x+3a=\lambda^2+\lambda\mu+\mu^2$

$$\text{and } t=(\lambda^3-\mu^3)/3\sqrt{2\mu}.$$

Now, t, z, or $\frac{1}{2}(r+r')$, y or $\frac{1}{2}\delta$, and therefore λ and μ, are all invariable; hence, in general, since area equals $\sqrt{2a\mu t}$, we have

$$\text{Area of parabolic sector}=\sqrt{a}(\lambda^3-\mu^3)/3,$$

where $2\lambda^2=r+r'+\delta$, and $2\mu^2=r+r'-\delta$.

Note. The sign of μ changes when PQ cuts AS produced.

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MONISM AND THE ANTINOMIES.

In the following sketch it is the purpose of the author to offer a few brief suggestions with a view to clearing up some of the long-mooted problems of philosophy and theology, particularly in regard to the so-called philosophical antinomies. Of these Kant has enumerated four, viz., (1) whether the universe is temporal and finite, or eternal and infinite; (2) whether composite substances are capable of resolution into irresolvable simple parts, or whether "naught that is simple exists"; (3) whether causality is the sole cause of phenomena or whether free will must be considered along with it; and (4) whether the world (i. e., the sum-totality of existence) possesses or does not possess some form of necessary existence. The contention of Kant is that these paradoxes do not exist, save in our own thought, and that it is due to the limitation of our mental powers that we seem to see such antinomies. With regard to the most important of these, namely, whether causality is the sole cause of existence, he has demonstrated absolutely, and with